June 25, 1999

Finding and solving Calogero-Moser type systems using Yang-Mills gauge theories

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Abstract

Yang-Mills gauge theory models on a cylinder coupled to external matter charges provide powerful means to find and solve certain non-linear integrable systems. We show that, depending on the choice of gauge group and matter charges, such a Yang-Mills model is equivalent to trigonometric Calogero-Moser systems and certain known spin generalizations thereof. Choosing a more general ansatz for the matter charges allows us to obtain and solve novel integrable systems. The key property we use to prove integrability and to solve these systems is gauge invariance of the corresponding Yang-Mills model.

1 Introduction

In a previous paper [1] we presented a novel class of integrable spin–particle systems which contains known integrable systems of Calogero-Moser (CM) type [2] and certain known spin generalization [3, 4] thereof as special cases, and many other generalizations which (to our knowledge) were not known before. Our method not only allowed us to find and prove integrability of these models but also to solve them explicitly. Subsequently two alternative derivations of these models also proving integrability were given by Polychronakos [5, 6].

In the present paper we give a more detailed account of our approach and also present extensions of our previous results. We made some effort to make the paper easily accessible to different readers: those who quickly want to get the flavor of how our method work, but also those who are interested in the details. In the rest of this

section we give some introduction to CM type systems, describe our method, and then give a plan for the paper.

About thirty years ago it was discovered that a dynamical system of particles on the line described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{\alpha=1}^{N}(p^\alpha)^2 + \frac{e^2}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} v(q^\alpha - q^\beta) \tag{1}$$

is completely integrable when the pair potential $v(r)$ equals a Weierstrass elliptic function $\wp(r)$, important special cases of which are $1/r^2$, $a^2/\sin^2(ar)$, and $a^2/\sinh^2(ar)$ [2] (for an early review see [7]). Subsequently these models have received much interest in different contexts, and various generalizations of these models have been found and studied (for recent reviews see, e.g., [8, 9]). Recently it was observed that these models can be obtained from Yang-Mills gauge theories on the cylinder coupled to particular non-dynamical matter charges [10].[1] Exploring this relation further we found and solved a large class of novel integrable spin–particle systems [1]. In this paper we give a more detailed account of these and some new results. The key property we use to prove integrability and to solve these systems is gauge invariance of the corresponding Yang-Mills model. This allows us to use different gauges, i.e., to impose different constraints compatible with gauge invariance. There is a gauge in which the dynamics of such a Yang-Mills model is equivalent to the dynamics of a CM type system, whereas in another gauge the dynamics is free and the solution can be found trivially. Thus the solution of the former system can be obtained from the latter solution by constructing a certain gauge transformation.

Relating a CM type system to a Yang-Mills theory makes integrability obvious and the construction of conservation laws (nearly) trivial. Moreover, this relation provides a rather simple method to construct an explicit solution. (This method can be regarded as an extension of the projection method [7].) Since there is a large freedom in choosing the external matter charges, one can obtain and solve a large number of different integrable systems. We believe that it should be possible to find other integrable models using our method, for example by considering more general gauge groups etc.

The plan of this paper is as follows. In Section 2 we summarize the facts about Yang-Mills gauge theories on a cylinder which we need in the sequel. The derivation of a certain class of dynamical systems from these gauge theories is explained in

[1] This is actually implicit already in earlier work; see, e.g., [11]

Section 3. We present two different arguments: the first argument (Section 3.1) is quick but only heuristic, and the second (Section 3.2) is somewhat less intuitive but rigorous. In Section 4 we show how to exploit the relation of these dynamical systems to gauge theories to explicitly solve interesting special cases of these systems, and we also show how to obtain Lax pairs and conservation laws in our formalism. As a warm–up, we first show how known results about CM models with potentials $1/r^2$, $a^2/\sin^2(ar)$ and $a^2/\sinh^2(ar)$ can be recovered (Sections 4.1–4.3). We will also refer to the former case as *Calogero model* and the latter as *Sutherland model*. We then derive and extend the results for the novel systems found in [1] (Section 4.4). A novel class of models which can not be solved in such an explicit manner but still should be integrable is discussed in Section 4.5.

Readers who only want to get a flavor of how our methods work are advised to read the beginning of Section 4 and proceed to Sections 4.2 and 4.4.

Notation. We denote as gl_N the complex $N \times N$ matrices, GL_N the complex invertible $N \times N$ matrices, and I the $N \times N$ unit matrix.

2 Yang-Mills theory on a cylinder

In this section we fix notation and define the gauge theory models of interest for our purposes. We also discuss different gauge conditions which will play important roles in the sequel.

2.1 Definition

We consider Yang-Mills theory on a cylinder with external matter sources. We restrict ourselves to a two-dimensional spacetime which is a cylinder, i.e., the time coordinate is $t \in \mathbb{R}$, and space is a circle parameterized by $x \in [-\pi, \pi]$. In the following, $\mu, \nu \in \{0, 1\}$ are spacetime indices, and repeated spacetime indices are summed over.[2] Our metric tensor is $\mathrm{diag}(1, -1)$.

Our starting point is the Lagrangian

$$\mathcal{L}(t,x) = -\frac{1}{2\pi}\left(\frac{1}{4}tr\left[F_{\mu\nu}(t,x)F^{\mu\nu}(t,x) + A_\nu(t,x)j^\nu(t,x)\right]\right) \tag{2}$$

[2] Note that we will use this summation convention *only* for spacetime indices.

where we set

$$j^1 \equiv 0, \quad j^0 \equiv \rho. \tag{3}$$

This Lagrangian describes a dynamical Yang-Mills field A_ν coupled to an external matter current j^ν whose spatial component vanishes. We shall denote the temporal component $j^0 = \rho$ of this matter current as *charge*. The Yang-Mills curvature $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \tag{4}$$

Note that the only non-trivial component of this is

$$E(t,x) := F_{01}(t,x) = -F_{10}(t,x). \tag{5}$$

We assume a representation of the structure groups such that A_μ and ρ are gl_N-valued functions.

2.2 Gauges

The Lagrangian defined in Eqs. (2) and (3) is obviously invariant under all transformations

$$\begin{aligned} A_\mu(t,x) &\to A_\mu^U(t,x) = U^{-1}(t,x)A_\mu(t,x)U(t,x) + \frac{1}{ig}U^{-1}(t,x)\partial_\mu U(t,x) \\ \rho(t,x) &\to \rho^U(t,x) = U^{-1}(t,x)\rho(t,x)U(t,x) \\ E(t,x) &\to E^U(t,x) = U^{-1}(t,x)E(t,x)U(t,x) \end{aligned} \tag{6}$$

for all differentiable GL_N-valued functions U on spacetime (the last relation is of course not independent since it follows from the first). This implies that the equations of motion derived from the Lagrangian (2)–(3) are also invariant under all these transformations. We will consider functions $U(t,x)$ which are differentiable only *locally* in time, i.e., it is differentiable for t in some non-empty interval (t_1, t_2) and all $x \in [-\pi, \pi]$. We denote the set of all such functions as $\mathcal{G}_{(t_1,t_2)}$. We refer to the transformations in Eq. (6) as *gauge transformations*. *Gauge invariance*, i.e., invariance under all gauge transformations, will allow us to impose additional constraints on the gauge fields. We call these constraints *gauges*. There are two different gauges which will be important for us.

Diagonal Coulomb Gauge

This gauge is defined by the condition that A_1 is a diagonal matrix which is independent of x, i.e.,

$$A_1(t,x) = Q(t) = \mathrm{diag}\left(q^1(t), q^2(t), \ldots, q^N(t)\right). \tag{7}$$

To show that this indeed is a gauge one has to prove that it is always possible to find a gauge transformation bringing the Yang-Mills field into the form of Eq. (7), i.e., for each (generic) Yang-Mills configuration $A_1(t,x)$ one can find a gauge transformation U such that $A_1^U(t,x)$ defined by Eq. (6) has the form of $Q(t)$ in Eq. (7). For completeness we now recall a proof of this by explicit construction of the required U (see [1] and references therein).

We first note that the condition $A_1^U = Q$ is equivalent to the differential equation $\partial_1 U + igA_1U = igUQ$. To solve this equation we first consider the boundary value problem

$$\partial_1 S(t,x) + igA_1(t,x)S(t,x) = 0, \quad S(t,-\pi) = I. \tag{8}$$

This problem has a unique solution (see, e.g., Ref. [13], Theorems 1.1 and 2.1) which we write as

$$S(t,x) = \mathcal{P}\exp\left(-ig\int_{-\pi}^{x} dy\, A_1(t,y)\right) \tag{9}$$

where the symbol $\mathcal{P}\exp$ denotes the path ordered exponential.[3] It is important to note that $S(t,x)$ is not a gauge transformation since it is not periodic in x (its values at $x=-\pi$ and π are different in general). However, it can be used to construct a gauge transformation as follows. We first introduce an important technical condition. We call a gauge field $A_1(t,x)$ *regular* (at time t) if the corresponding matrix $S(t,\pi)$ is non-degenerate, i.e., all the eigenvalues of $S(t,\pi)$ are different. If $A_1(t,x)$ is regular then there exists an invertible matrix $V(t)$ diagonalizing $S(t,\pi)$ (see e.g. Theorem 10.2.4. in [14]), i.e.,

$$V(t)^{-1}S(t,\pi)V(t) = e^{-ig2\pi Q(t)} \tag{10}$$

for some diagonal matrix $Q(t) = \mathrm{diag}(q^1(t),\ldots,q^N(t))$. We now claim that the function

$$U(t,x) = S(t,x)V(t)e^{ig(x+\pi)Q(t)} \tag{11}$$

is periodic. Indeed,

$$U(t,\pi) = V(t)V(t)^{-1}S(t,\pi)V(t)e^{ig2\pi Q(t)} = V(t) = U(t,-\pi)$$

[3] in the terminology of Ref. [13], $S(t,x)$ is identical with the product integral $\prod_{-\pi}^{x} e^{-igA_1(t,s)ds}$

where we inserted $V(t)V(t)^{-1} = I$ and used Eq. (10). Moreover, U Eq. (11) satisfies $\partial_1 U + igA_1 U = igUQ$ equivalent to $A_1^U = Q$. It is also easy to see that $U(t,x)$ is invertible (see e.g. Ref. [13], Section 1.1). We are left to show that $U(t,x)$ is a differentiable function on spacetime. This might seem trivial but is actually not since $V(t)$ and $Q(t)$ as defined in Eq. (10) can be discontinuous in t; see, e.g., [15]. However, if $A_1(t,x)$ is regular for all t in an open time interval (t_1,t_2) then $Q(t)$ and $V(t)$ can be chosen to be differentiable in this time interval [15], and $U(t,x)$ in Eq. (11) is indeed a differentiable function, i.e., a gauge transformation in $\mathcal{G}_{(t_1,t_2)}$. We call a Yang-Mills field $A_1(t,x)$ *generic* if it is regular for all times $t \in \mathbb{R}$.

It is worth noting that the function $V(t)$ is not unique: all transformations

$$V(t) \to V'(t) = V(t)D(t), \quad D(t) = \mathrm{diag}(d^1(t), \ldots, d^N(t)) \tag{12}$$

with arbitrary differentiable and non-zero functions $d^\alpha(t)$, $\alpha = 1, 2, \ldots, N$, are compatible with the conditions determining $V(t)$. This corresponds to the residual gauge freedom which remains after imposing the diagonal Coulomb gauge: gauge transformations Eq. (6) with $U(t,x) = D(t)$ (diagonal and independent of x) leave the diagonal Coulomb gauge Eq. (7) invariant but act non-trivially on A_0 and ρ.

We note that the Eqs. (8) and (10) provide the recipe how to compute $Q(t)$ from a given Yang-Mills configuration $A_1(t,x)$. This will play an important role for us in the sequel.

Remark: We note that in our examples later the eigenvalues $e^{-ig2\pi q^\alpha(t)}$ of $S(t,\pi)$ have the following physical interpretation: $q^\alpha(t)$, $\alpha = 1, \ldots, N$, correspond to the positions of interacting particles in a dynamical system. The technical condition of a Yang-Mills field being regular in some interval thus means that the particles do not collide with each other. Since the particle interactions are repulsive in all dynamical systems we encounter, it is plausible that regularity holds for all times, but we will be able to prove this only in certain special cases. In general we can prove our results only locally in time, i.e., for time intervals where no particle collisions occur.

Weyl gauge

This gauge is defined by the condition

$$A_0(t,x) = 0. \tag{13}$$

To show that this is a gauge one can use a similar argument as above, i.e., for a given $A_0(t,x)$, the function

$$W(t,x) = \mathcal{P}\exp\left(-ig\int_0^t dt'\, A_0(t',x)\right), \tag{14}$$

is such that $A_0^W(t,x)$ Eq. (6) vanishes. Note that $W(t,x)$ here is indeed a gauge transformation (i.e., a periodic, invertible, and differentiable function on spacetime). We note that if our spacetime was a torus and not a cylinder, the Weyl- and the diagonal Coulomb gauge would be more similar: one of them is obtained from the other by interchanging the space- and time coordinates x and t. The reason why the Weyl gauge is simpler on the cylinder is that there is no periodicity condition in t.

3 Dynamical systems from Yang-Mills theories on the cylinder

In this section we show how to derive certain dynamical systems from Yang-Mills systems on a cylinder with non-dynamical external matter charges. We will use two different arguments which both lead to the same results. In Section 3.1 we use a canonical procedure to formulate our Yang-Mills system as an infinite dimensional Hamiltonian system with a constraint called Gauss' law. By solving Gauss' law in the diagonal Coulomb gauge Eq. (7) we obtain an integrable non-linear Hamiltonian system which generalizes the Calogero- and Sutherland models discussed in Section 1. This Hamiltonian method is conceptually simple but not quite satisfactory from a mathematical point of view. To make it mathematically rigorous would require a deeper analysis which is beyond the scope of the present paper. Instead, we discuss an alternative and complimentary method in Section 3.2. This method uses only the equations of motion and thus circumvents all the mathematical difficulties which one would have to face in a rigorous discussion of the method in Section 3.1.

3.1 Hamiltonian approach

In the following it is useful to denote the elements of the matrices $M = A_\mu, E, \rho$ by $M^{\alpha\beta}$, $\alpha,\beta = 1,2,\ldots N$, and tr the usual $N\times N$ matrix trace. Noting that $tr(MM') = \sum_{\alpha,\beta=1}^N M^{\alpha\beta}(M')^{\beta\alpha}$ and following the standard canonical procedure [16] we obtain from the Lagrangian (2)–(3) (up to a surface term which vanishes due to

the periodicity of $A_0 E$ in x)

$$\mathcal{H} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \left[tr\left(\frac{1}{2}E^2 - A_0[\partial_1 E + ig[A_1, E] - \rho]\right)\right]. \tag{15}$$

The variable conjugate to $A_1^{\alpha\beta}(x)$ is $E^{\beta\alpha}(x)$, which implies the Poisson brackets

$$\{A_1^{\alpha\beta}(x), E^{\alpha'\beta'}(y)\} = 2\pi\delta^{\alpha\beta'}\delta^{\beta\alpha'}\delta(x-y). \tag{16}$$

Since the Lagrangian is independent of $A_0^{\alpha\beta}(x)$ its conjugate momentum, which we denote as $\Pi_0^{\beta\alpha}(x)$, has to vanish: $\Pi_0 \simeq 0$. Here '$\simeq$' means that this is to be regarded as a constraint [16]. This primary constraint implies the secondary constraint $\partial_0\Pi_0 = \{\Pi_0, H\} \simeq 0$, i.e.,

$$G(x) := \partial_1 E(x) + ig[A_1(x), E(x)] - \rho(x) \simeq 0, \tag{17}$$

which is called *Gauss' law*. The tertiary constraint $\partial_0 G(x) = \{G(x), H\} \simeq 0$ together with $\{A_1(x), \rho(y)\} = \{E(x), \rho(y)\} = 0$ then fixes the Poisson bracket of the charges (see Appendix B for details)

$$\{\rho^{\alpha\beta}(x), \rho^{\alpha'\beta'}(y)\} \simeq ig2\pi\left[\rho^{\alpha\beta'}(x)\delta^{\beta\alpha'} - \rho^{\alpha'\beta}(x)\delta^{\beta'\alpha}\right]\delta(x-y). \tag{18}$$

Note that,

$$\{G^{\alpha\beta}(x), G^{\alpha'\beta'}(y)\} \simeq ig2\pi\left[G^{\alpha\beta'}(x)\delta^{\beta\alpha'} - G^{\alpha'\beta}(x)\delta^{\beta'\alpha}\right]\delta(x-y),$$

and the Poisson brackets above also fix the time evolution of the charges, $\partial_0\rho(x) = \{\rho(x), H\}$.

We now exploit the gauge freedom and impose the gauges discussed in the last section. We shall do that by replacing '$\simeq$' by '$=$' in the equations given above, i.e., we ignore possible subtleties and treat constraints like strict equalities. We therefore regard the following as a simple but only heuristic argument, as discussed above.

Diagonal Coulomb Gauge

As shown in the last section, it is always possible to transform A_1 to a x-independent diagonal matrix. This means that the Yang-Mills field only has a finite number of true dynamical degrees of freedom. In order to obtain a Hamiltonian for these degrees of freedom only, we impose the gauge condition Eq. (7). We then use Gauss' law to eliminate all but the true dynamical degrees of freedom.

To do this we will use Fourier transformations

$$\hat{E}^{\alpha\beta}(n) = \int_{-\pi}^{\pi} dx\, e^{-inx} E^{\alpha\beta}(x), \quad n \in \mathbb{Z} \tag{19}$$

and similarly for A_1 and ρ. Using this and imposing Eq. (7), the Gauss' law can be rewritten in component form as

$$i\left(n + g[q^\alpha - q^\beta]\right)\hat{E}^{\alpha\beta}(n) = \hat{\rho}^{\alpha\beta}(n). \tag{20}$$

For $\alpha = \beta$ and $n = 0$, (20) implies

$$\hat{\rho}^{\alpha\alpha}(0) = 0. \tag{21}$$

In all other cases, Eq. (20) can be solved for $\hat{E}^{\alpha\beta}(n)$. Inserting these in the Hamiltonian (15), we obtain

$$\begin{aligned} \mathcal{H} &= \frac{1}{4\pi}\int_{-\pi}^{\pi} dx\, tr(E(x)^2) = \frac{1}{8\pi^2}\sum_{n\in\mathbb{Z}}\sum_{\alpha,\beta=1}^{N} \hat{E}^{\alpha\beta}(n)\hat{E}^{\beta\alpha}(-n) = \\ &= \sum_{\alpha=1}^{N}\frac{\hat{E}^{\alpha\alpha}(0)\hat{E}^{\alpha\alpha}(0)}{2(2\pi)^2} + \frac{1}{8\pi^2}\sum_{n\in\mathbb{Z}}\sum_{\alpha,\beta=1}^{N}(1-\delta_{n,0}\delta^{\alpha\beta})\frac{\hat{\rho}^{\alpha\beta}(n)\hat{\rho}^{\beta\alpha}(-n)}{(n+g[q^\alpha - q^\beta])^2}. \end{aligned}$$

Here a comment on our notation is in order: for $\alpha = \beta$ and $n = 0$ the last term seems ambiguous since it formally is $0/0$. However, from our derivation above it is clear that this has to be interpreted as 0, i.e., here and in the following

$$(1-\delta_{n,0}\delta^{\alpha\beta})(\cdots) \equiv 0 \quad \text{if } n = 0 \text{ and } \alpha = \beta$$

even if $(\cdots)$ happens to be infinite.

We now find it convenient to introduce the following suggestive notation,

$$p^\alpha = \frac{\hat{E}^{\alpha\alpha}(0)}{2\pi}. \tag{22}$$

Moreover, we will see later that the functions $\hat{A}_0^{\alpha\alpha}(t,0) \equiv a^\alpha(t)$ play a special role and can be chosen arbitrarily. Thus despite of the constraint Eq. (21) we can leave the corresponding term $\propto \sum_\alpha a^\alpha(t)\hat{\rho}^{\alpha\alpha}(t, n=0)$ in the Hamiltonian. We thus obtain

$$\mathcal{H} = \sum_{\alpha=1}^{N}\frac{(p^\alpha)^2}{2} + \frac{1}{8\pi^2}\sum_{n\in\mathbb{Z}}\sum_{\alpha,\beta=1}^{N}(1-\delta_{n,0}\delta^{\alpha\beta})\frac{\hat{\rho}^{\alpha\beta}(n)\hat{\rho}^{\beta\alpha}(-n)}{(n+g[q^\alpha-q^\beta])^2} + \frac{1}{4\pi^2}\sum_{\alpha=1}^{N} a^\alpha \hat{\rho}^{\alpha\alpha}(0) \tag{23}$$

where $a^\alpha(t)$ play the role of an arbitrary external time dependent field which contribute to the time evolution of $\hat{\rho}^{\alpha\beta}(n)$.

Using Eq. (16) it is easy to verify that

$$\{q^\alpha, p^\beta\} = \left\{ A_1^{\alpha\alpha}(x), \frac{\hat{E}^{\beta\beta}(0)}{2\pi} \right\} = \left\{ A_1^{\alpha\alpha}(x), \frac{\int_{-\pi}^{\pi} E^{\beta\beta}(y)dy}{2\pi} \right\} = \delta^{\alpha\beta}. \tag{24}$$

Hence, it is natural to interpret p^α as particle momenta and q^α as the corresponding position variables. From the Poisson bracket (18) we obtain the corresponding Poisson bracket for $\hat{\rho}$,

$$\{\hat{\rho}^{\alpha\beta}(n), \hat{\rho}^{\alpha'\beta'}(m)\} = ig2\pi[\hat{\rho}^{\alpha\beta'}(n+m)\delta^{\beta\alpha'} - \hat{\rho}^{\alpha'\beta}(n+m)\delta^{\beta'\alpha}]. \tag{25}$$

We interpret the $\hat{\rho}^{\alpha\beta}(n)$ as spin degrees of freedom. Hence, we have obtained a dynamical system described by the Hamiltonian (23), together with the Poisson brackets (24) and (25). All other Poisson brackets vanish.

It is interesting to note that the Hamiltonian Eq. (23) is invariant under the transformations

$$q^\alpha \to q^\alpha + \frac{m^\alpha}{g}, \quad p^\alpha \to p^\alpha, \quad \hat{\rho}^{\alpha\beta}(n) \to \hat{\rho}^{\alpha\beta}(n + m^\alpha - m^\beta) \tag{26}$$

for all integers m^α. Thus, if q^α, $\alpha = 1, \ldots, N$, are real, this model describes particles moving on a circle of length $1/g$ and interacting with a potential whose strength depends on the spin degrees of freedom. Moreover, this Hamiltonian is invariant under all permutations of the particle labels. It is worth noting that the existence of these symmetry transformations is due to the Gribov ambiguities of the diagonal Coulomb gauge as discussed in Ref. [17].

Using the Hamiltonian Eq. (23) and the Poisson brackets given above it is straightforward to derive the equations of motion $\partial_0 X = \{X, \mathcal{H}\}$ for $X = q^\alpha, p^\alpha$ and $\hat{\rho}^{\alpha\beta}(n)$. (We will write down these equations in Section 3.2.) Note that these equations are highly non-linear.

Weyl gauge

In the Weyl gauge, the Hamiltonian is simply

$$\mathcal{H} = \frac{1}{4\pi} \int_{-\pi}^{\pi} dx \, tr(E(x)^2) \tag{27}$$

which shows that we have a free (non-interacting) system. Still, the system is not trivial since we have to account for Gauss' law Eq. (17). However, it is easy to see that all quantities $tr[E(x)^n]$ (arbitrary $x \in [-\pi, \pi]$ and $n \in \mathbb{N}$) are conserved in time t. Thus there is an infinite number of conservation laws. (Note that we do not claim that these conservation laws are independent.) Since the system is free, we can actually solve the system in the Weyl gauge (this will be done in the next section).

3.2 Alternative approach based on equations of motion

Our discussion in the last section suggests that our gauge theory is an integrable system and equivalent to a non-linear dynamical system which is obtained by imposing the diagonal Coulomb gauge. Instead of trying to make the argument in the last section rigorous (which would be interesting but is beyond the scope of the present paper) we now present an alternative, less intuitive, argument leading to the same conclusions but avoiding the mathematical difficulties of the canonical procedure.

We start with the Lagrangian equation obtained from the Lagrangian (2). They can be written as (recall that our metric tensor used to rise and lower spacetime indices is $\mathrm{diag}(1, -1)$, and that we use a summation convention for spacetime indices $\mu, \nu \in \{0, 1\}$)

$$[D_\mu, F^{\mu\nu}] = j^\nu, \quad F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] \tag{28}$$

with $j^1 = 0$ and $j^0 = \rho$. Here we introduced the covariant derivative

$$D_\mu = \partial_\mu + igA_\mu. \tag{29}$$

In this paragraph (only!) we use a notation which takes into account the Leibniz rule for differentiation. In this notation one distinguishes, e.g., $(\partial_\nu A_\mu)$ from $\partial_\nu A_\mu = (\partial_\nu A_\mu) + A_\mu \partial_\nu$ which is obtained by using the Leibniz rule. Thus $(\partial_\nu A_\mu) = [\partial_\nu, A_\mu]$. In this notation, gauge transformations Eq. (6) can be simply written as $D_\mu \to U^{-1} D_\mu U$ and $j_\mu \to U^{-1} j_\mu U$, and it is thus obvious that the Lagrangian equations above are gauge invariant. These equations imply

$$[D_\nu, j^\nu] = [D_\nu, [D_\mu, F^{\mu\nu}]] = ig[F_{\mu\nu}, F^{\mu\nu}] + [D_\mu, [D_\nu, F^{\mu\nu}]] = -[D_\mu, j^\mu]$$

where we used the Jacobi identity, the anti-symmetry for the commutator, and $[F_{\mu\nu}, F^{\mu\nu}] = 0$ (cf. Eq. (5)). We thus also have the equation $[D_\nu, j^\nu] = [D_0, \rho] = 0$.

We can now forget that these differential equations originate from a Lagrangian and instead use them as definition of our Yang-Mills model. We find it convenient

to write these equations in the following form (we use the definition in Eq. (5); to simplify notation we write $(\partial_\mu X)$ as $\partial_\mu X$ in the sequel),

$$\partial_0 A_1 = E + \partial_1 A_0 + ig[A_1, A_0] \tag{30}$$
$$\partial_0 E + ig[A_0, E] = 0 \tag{31}$$
$$\partial_0 \rho + ig[A_0, \rho] = 0 \tag{32}$$
$$\partial_1 E + ig[A_1, E] = \rho. \tag{33}$$

Note that gauge invariance trivially implies the important

Proposition: *If $A_\mu(t,x), E(t,x)$ and $\rho(t,x)$ is a solution to the Eqs. (30)–(33), then $A^U_\mu(t,x), E^U(t,x)$ and $\rho^U(t,x)$ Eq. (6) is also a solution for arbitrary differentiable GL_N-valued functions $U(t,x)$ on spacetime.*

We now rewrite these equations in our two different gauges.

Diagonal Coulomb Gauge

As argued in Section 2, it is always possible to find a gauge transformation (given by Eq. (11)) bringing A_1 into the form Eq. (7). We now impose this gauge, i.e., we replace A_μ, E and ρ by A^U_1, E^U and ρ^U with U given by Eq. (11). Using Fourier transformation, Eqs. (30)–(33) can be written as (to simplify notation, we still use the symbols E, A_0, ρ instead of E^U, A^U_0, ρ^U)

$$2\pi\delta_{n,0}\delta^{\alpha\beta}\partial_0 q^\alpha(t) = \hat{E}^{\alpha\beta}(t,n) + i(n + g[q^\alpha(t) - q^\beta(t)])\hat{A}^{\alpha\beta}_0(t,n), \tag{34}$$

$$\partial_0 \hat{E}^{\alpha\beta}(t,n) = -\frac{ig}{2\pi}\sum_{k\in\mathbb{Z}}\sum_{\gamma=1}^{N}\left(\hat{A}^{\alpha\gamma}_0(t,k)\hat{E}^{\gamma\beta}(t,n-k) - \hat{E}^{\alpha\gamma}(t,k)\hat{A}^{\gamma\beta}_0(t,n-k)\right), \tag{35}$$

$$\partial_0 \hat{\rho}^{\alpha\beta}(t,n) = -\frac{ig}{2\pi}\sum_{k\in\mathbb{Z}}\sum_{\gamma=1}^{N}\left(\hat{A}^{\alpha\gamma}_0(t,k)\hat{\rho}^{\gamma\beta}(t,n-k) - \hat{\rho}^{\alpha\gamma}(t,k)\hat{A}^{\gamma\beta}_0(t,n-k)\right), \tag{36}$$

$$\hat{\rho}^{\alpha\beta}(t,n) = i(n + g[q^\alpha(t) - q^\beta(t)])\hat{E}^{\alpha\beta}(t,n). \tag{37}$$

Note that these equations determine all components of $\hat{A}^{\alpha\beta}_0(t,n)$ except for $\alpha = \beta$ and $n = 0$. We will use the following notation for the unspecified components,

$$\hat{A}^{\alpha\alpha}_0(t,0) := a^\alpha(t) \tag{38}$$

which are arbitrary functions. They correspond to the residual gauge freedom which is not fixed by the diagonal Coulomb gauge, as discussed after Eq. (12).

Eq. (34) yields

$$\hat{A}_0^{\alpha\beta}(t,n) = a^\alpha(t)\delta^{\alpha\beta}\delta_{n,0} + \left(1-\delta^{\alpha\beta}\delta_{n,0}\right)\frac{i\hat{E}^{\alpha\beta}(t,n)}{(n+g[q^\alpha(t)-q^\beta(t)])}, \tag{39}$$

and Eq. (37) implies

$$\hat{E}^{\alpha\beta}(t,n) = 2\pi p^\alpha(t)\delta^{\alpha\beta}\delta_{n,0} + \left(1-\delta^{\alpha\beta}\delta_{n,0}\right)\frac{-i\hat{\rho}^{\alpha\beta}(t,n)}{(n+g[q^\alpha(t)-q^\beta(t)])} \tag{40}$$

(we used the notation $\hat{E}^{\alpha\alpha}(t,0) = 2\pi p^\alpha(t)$ introduced in Section 3.1). By combining these equations we obtain

$$\hat{A}_0^{\alpha\beta}(t,n) = a^\alpha(t)\delta^{\alpha\beta}\delta_{n,0} + \left(1-\delta^{\alpha\beta}\delta_{n,0}\right)\frac{\hat{\rho}^{\alpha\beta}(t,n)}{(n+g[q^\alpha(t)-q^\beta(t)])^2}. \tag{41}$$

We now use these relations to derive the time evolution of our dynamical quantities in the diagonal Coulomb gauge. Putting $\beta = \alpha$ and $n = 0$ in Eq. (34) and using again $\hat{E}^{\alpha\alpha}(t,0) = 2\pi p^\alpha(t)$ we obtain

$$\partial_0 q^\alpha(t) = p^\alpha(t). \tag{42}$$

The time evolution of the $p^\alpha(t) = \hat{E}^{\alpha\alpha}(t,0)/2\pi$ follows by inserting Eqs. (40) and (41) in Eq. (35):

$$\partial_0 p^\alpha(t) = \frac{2g}{(2\pi)^2}\sum_{n\in\mathbb{Z}}\sum_{\gamma=1}^{N}(1-\delta^{\alpha\gamma}\delta_{n,0})\frac{\hat{\rho}^{\alpha\gamma}(t,n)\hat{\rho}^{\gamma\alpha}(t,-n)}{(n+g[q^\alpha(t)-q^\gamma(t)])^3}. \tag{43}$$

Inserting Eq. (38) and (41) in (36) we get the time evolution of $\hat{\rho}^{\alpha\beta}(t,n)$

$$\begin{aligned}
\partial_0\hat{\rho}^{\alpha\beta}(t,n) &= -\,-\frac{ig}{2\pi}\hat{\rho}^{\alpha\beta}(t,n)\left(a^\alpha(t)-a^\beta(t)\right) - \\
&\quad -\frac{ig}{2\pi}\sum_{k\in\mathbb{Z}}\sum_{\gamma=1}^{N}\Bigg[(1-\delta^{\alpha\gamma}\delta_{k,0})\frac{\hat{\rho}^{\alpha\gamma}(t,k)\hat{\rho}^{\gamma\beta}(t,n-k)}{(k+g[q^\alpha(t)-q^\gamma(t)])^2} - \\
&\quad -\left(1-\delta^{\gamma\beta}\delta_{k,0}\right)\frac{\hat{\rho}^{\alpha\gamma}(t,n-k)\hat{\rho}^{\gamma\beta}(t,k)}{(k+g[q^\gamma(t)-q^\beta(t)])^2}\Bigg]
\end{aligned} \tag{44}$$

We now have obtained the time evolution equations for our dynamical system in a direct way. This result justifies our derivation of the Hamiltonian (23) in the last section: *The equations (42)–(44) are identical with the Hamiltonian equations obtained from the Hamiltonian (23) and the Poisson brackets (24)–(25).* (The proof of this is a straightforward computation which we skip.)

It is worth to note that p^α and q^α are not necessarily real. Denoting the real and imaginary parts of p^α by p_r^α and p_i^α, respectively, and similarly for q^α, we obtain from Eq. (42)

$$\partial_0 q_r^\alpha(t) = p_r^\alpha(t),$$

and from Eq. (43)

$$\partial_0 p_r^\alpha(t) = \mathrm{Re}\left[\frac{2g}{(2\pi)^2}\sum_{n\in\mathbb{Z}}\sum_{\gamma=1}^{N}(1-\delta^{\alpha\gamma}\delta_{n,0})\frac{\hat\rho^{\alpha\gamma}(t,n)\hat\rho^{\gamma\alpha}(t,-n)}{(n+g[q_r^\alpha(t)-q_r^\gamma(t)+i[q_i^\alpha(t)-q_i^\gamma(t)]])^3}\right]$$

and similarly for the imaginary parts. Similarly, the time evolution of Eq. (44) can be split in real and imaginary parts. In this general case, we can interpret the model as describing $2N$ particles interacting with each other and spin degrees of freedom.

Weyl gauge

We now discuss our dynamical system in the Weyl gauge Eq. (13). Imposing this condition, (30)–(33) are equivalent to

$$\begin{aligned}\partial_0 A_1 = E, \quad \partial_0 E = 0, \quad \partial_0\rho = 0\\ \partial_1 E + ig[A_1, E] = \rho.\end{aligned} \tag{45}$$

which can be solved trivially (see Section 4). We also observe that *these equations are equivalent to the Hamiltonian equations obtained from the Hamiltonian (27) and the Poisson brackets (16) and (18).* (Again the proof is a simple computation which we skip.) We thus have rigorously justified all results which in Section 3.1 using the Hamiltonian method.

4 Solvable non-linear systems

In this section we discuss how to obtain and solve dynamical systems which are special cases of the system defined by Eqs. (42)–(44). We will impose additional constraints

on the charges ρ in order to obtain simpler systems which have natural physical interpretations and which can be solved explicitly. This will be done according to the following recipe:

1. Make an ansatz for the charge $\rho(t,x)$ which is parameterized by a finite number of x-independent functions and which is consistent with Eqs. (21) and (44).
2. Insert this ansatz for ρ in the Eqs. (42)–(44). To obtain the Hamiltonian and Poisson brackets giving rise to these equations, insert this ansatz for ρ in the Hamiltonian (23) and the Poisson bracket (25).

We will also show how to obtain a one-parameter family of Lax pairs and conservation laws in our formalism.

We are interested in the solution of the system of differential equations (42)–(44) with the initial conditions,

$$q^\alpha(0) = q_0^\alpha, \quad p^\alpha(0) = p_0^\alpha, \quad \hat{\rho}^{\alpha\beta}(0,n) = \hat{\rho}_0^{\alpha\beta}(n). \tag{46}$$

Our method of solving this initial value problem is based on the results obtained in the last section. It was shown there that the Eqs. (42)–(44) are obtained from a Yang-Mills gauge theory by imposing the diagonal Coulomb gauge Eq. (7). Moreover, this very gauge theory in the Weyl gauge Eq. (13) leads to the *linear* differential equations (45) which can be solved trivially,

$$\begin{aligned} E(t,x) &= E(0,x), \\ A_1(t,x) &= A_1(0,x) + E(0,x)t, \\ \rho(t,x) &= \rho(0,x) \end{aligned} \tag{47}$$

with the initial data $E(0,x)$, $A_1(0,x)$ and $\rho(0,x)$ satisfying Gauss' law Eq. (33). It is important to note that our solution (47) satisfies the Gauss' law for all t if it satisfies it for $t = 0$. From our discussion in the last section it is obvious that initial conditions for the Yang-Mills fields corresponding to the initial conditions Eq. (46) are,

$$A_1^{\alpha\beta}(0,x) = \delta^{\alpha\beta} q_0^\alpha, \quad \int_{-\pi}^{\pi} dx E^{\alpha\alpha}(0,x) = 2\pi p_0^\alpha, \quad \hat{\rho}^{\alpha\beta}(0,n) = \hat{\rho}_0^{\alpha\beta}(n). \tag{48}$$

Gauss' law Eq. (33) then completely determines $E^{\alpha\beta}(0,x)$ and thus the solution of our gauge theory model in the Weyl gauge. In Section 2.2, we also discussed how to construct the gauge transformation transforming to the diagonal Coulomb gauge. Combining these results we obtain the following recipe for solving a large class of non-linear integrable systems:

1. Use the initial conditions Eq. (48) to calculate $E(0,x)$ from the Gauss' law Eq. (33).

2. Take $A_1(t,x)$ given by Eq. (47) and calculate $S(t,\pi)$ from Eq. (8).

3. The positions $q^\alpha(t)$ at time t are given by the eigenvalues $\lambda^\alpha(t)$ of $S(t,\pi)$ according to $\lambda^\alpha(t) = e^{-ig2\pi q^\alpha(t)}$, provided that $\lambda^\alpha(t') \neq \lambda^\beta(t')$ for all $\alpha \neq \beta$ and $0 \leq t' \leq t$. If this latter condition holds we say that *no particle collisions occur in the time interval* $[0,t]$.

4. The solutions $\rho(t,x)$ are given by $U^{-1}(t,x)\rho(0,x)U(t,x)$ where $U(t,x)$ is the gauge transformation defined in Eq. (11).

We note that the no-collision condition above is due to the regularity assumption which we needed to prove the existence of the diagonal Coulomb gauge in Section 2.2.

We now show how to obtain Lax equations and conservation laws in our formalism. For that we observe that one of our Yang-Mills equations, $\partial_0 E(t,x) + ig[A_0(t,x), E(t,x)] = 0$, has the form of a Lax equation for arbitrary fixed x: For any x, $L(t) \equiv E(t,x)$ and $M(t) \equiv igA_0(t,x)$ is a Lax pair, $\partial L/\partial t + [M,L] = 0$. We thus have a *one-parameter family of Lax pairs.* By a standard argument this implies that $tr[E(t,x)^n]$ is invariant under time evolution, for arbitrary positive integer n and $x \in [-\pi,\pi]$. We have already found these conservation laws by a different argument at the end of Section 3.1.

In case the computation here can be done explicitly we obtain explicit solutions, Lax pairs and conservation laws. We will now discuss several examples of increasing complexity where this is possible. Before that we state three identities which will be important for us.

Three identities. The first identity we will need is well-known,

$$\sum_{n\in\mathbf{Z}} \frac{1}{(n+r)^2} = \frac{\pi^2}{\sin^2(\pi r)}, \quad r \in \mathbb{C} \tag{49}$$

(see e.g. [12]). The second is a generalization of the first

$$\sum_{n\in\mathbb{Z}} \frac{e^{ins}}{(n+r)^2} = e^{-irs_{2\pi}} \left(\frac{\pi^2}{\sin^2(\pi r)} + i\pi s_{2\pi}\cot(\pi r) - \pi|s_{2\pi}| \right), \quad r \in \mathbb{C}, s \in \mathbb{R}, \tag{50}$$

where

$$s_{2\pi} := s - 2\pi n \quad \text{with } n \in \mathbb{Z} \text{ such that } -\pi \leq s - 2\pi n < \pi. \tag{51}$$

Finally,

$$\sum_{n \neq 0} \frac{e^{ins}}{n^2} = \frac{(s_{2\pi})^2}{2} - \pi |s_{2\pi}| + \frac{\pi^2}{3}, \quad s \in \mathbb{R}. \tag{52}$$

Proofs of Eqs. (50) and (52) are given in Appendix A.

4.1 Example 1: Spin CM model with $v(r) = r^{-2}$

As the simplest example we now show how to obtain and solve the Calogero model and certain spin generalizations thereof.

For that we make the following ansatz,

$$\hat{\rho}^{\alpha\beta}(t, n) = \delta_{n,0}(1 - \delta^{\alpha\beta}) 2\pi g s^{\alpha\beta}(t). \tag{53}$$

It is important to note that Eq. (44) implies that $\partial_0 \hat{\rho}^{\alpha\beta}(t, n) = 0$ for $n \neq 0$ and $\partial_0 \hat{\rho}^{\alpha\alpha}(t, 0) = 0$. Hence the ansatz Eq. (53) is consistent, and the functions $s^{\alpha\beta}(t)$ for $\alpha \neq \beta$, together with the $p^\alpha(t)$ and $q^\alpha(t)$, are the dynamical variables of a dynamical system.

From the Hamiltonian (23) and the Poisson brackets (24)–(25) we obtain

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \sum_{\alpha=1}^{N} (p^\alpha)^2 + \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{N} \frac{s^{\alpha\beta} s^{\beta\alpha}}{(q^\alpha - q^\beta)^2} && (54) \\ \{p^\alpha, q^\beta\} &= \delta^{\alpha\beta} && (55) \\ \{s^{\alpha\beta}, s^{\alpha'\beta'}\} &= i(s^{\alpha\beta'} \delta^{\beta\alpha'} - s^{\alpha'\beta} \delta^{\beta'\alpha}) && (56) \end{aligned}$$

which determines the equations of motion for the dynamical variables. Moreover, the initial conditions are

$$q^\alpha(0) = q_0^\alpha, \quad p^\alpha(0) = p_0^\alpha, \text{ and } s^{\alpha\beta}(0) = s_0^{\alpha\beta} \quad \text{all real.} \tag{57}$$

We now have to determine the corresponding initial conditions for the Yang-Mills fields. Using $E^{\alpha\beta}(0, x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{E}^{\alpha\beta}(0, n) e^{inx}$ and Eqs. (22) and (40) we obtain

$$E^{\alpha\beta}(0, x) = p_0^\alpha \delta^{\alpha\beta} + (1 - \delta^{\alpha\beta}) \frac{s_0^{\alpha\beta}}{i[q_0^\alpha - q_0^\beta]}. \tag{58}$$

Moreover, we recall $A_1^{\alpha\beta}(0,x) = \delta^{\alpha\beta} q_0^\alpha$. Inserting this and Eq. (58) in Eq. (47) we find that $A_1(t,x) = A_1(t)$ is independent of x,

$$A_1^{\alpha\beta}(t) = q_0^\alpha \delta^{\alpha\beta} + \left(p_0^\alpha \delta^{\alpha\beta} + (1-\delta^{\alpha\beta}) \frac{s_0^{\alpha\beta}}{i[q_0^\alpha - q_0^\beta]} \right) t. \tag{59}$$

This is the solution of the Yang-Mills equations. To find the solution for our dynamical system we need to transform this to the diagonal Coulomb gauge Eq. (7). This is simple here: all we need to do is diagonalize the matrix $A_1(t)$. Furthermore, our discussion in Section 4 implies that $s^{\alpha\beta}(t)$ can be obtained from the matrix $U(t)$ diagonalizing $A_1(t)$. We thus obtain the following

Result 1: *Provided that no particle collisions occur in the time interval* $[0,t]$*, the solution* $q^\alpha(t)$ *of the initial value problem corresponding to the Hamiltonian (54) and the Poisson brackets (55) and (56) is given by the eigenvalues of the matrix* $A_1(t)$ *in Eq. (59),*

$$U(t)^{-1} A_1(t) U(t) = \mathrm{diag}\left(q^1(t), q^2(t), \ldots, q^N(t) \right).$$

Moreover, the diagonalizing matrix $U(t)$ *gives the time evolution of the spin degrees of freedom,* $s(t) = U(t)^{-1} s(0) U(t)$*.*

We thus have recovered the integrable systems and their solution previously found in Ref. [4]. In the present case we only get one Lax pair (the Lax pair family is independent of x) which coincides with the one given in Ref. [4].

Note that the special case where all initial $s^{\alpha\beta}$ are identical, $s^{\alpha\beta}(0) = e$, corresponds to the Calogero model, i.e., the Hamiltonian Eq. (1) with $v(r) = r^{-2}$. To see this, it is important to notice that the functions $a^\alpha(t)$ in Eq. (44) can be chosen such that $s^{\alpha\beta}(t) = e$ for all t. Indeed, with

$$a^\alpha(t) = -\frac{2\pi e}{g} \sum_{\substack{\gamma=1 \\ \gamma\neq\alpha}}^{N} \frac{1}{(q^\alpha(t) - q^\gamma(t))^2} \tag{60}$$

we obtain $\partial_0 s^{\alpha\beta}(t) = 0$.

Remark: It would be interesting to find and study a case where collisions occur.

4.2 Example 2: CM model with $v(r) = a^2 \sin^{-2}(ar)$

In this section we show in detail how to recover the known solution of the Sutherland model, i.e., the Hamiltonian Eq. (1) with the potential $v(r) = a^2 \sin^{-2}(ar)$. This is a simple special case and warm-up for what is discussed in Section 4.4.

To obtain the Hamiltonian of the Sutherland model we make the ansatz

$$\hat{\rho}^{\alpha\beta}(0,n) = (1-\delta^{\alpha\beta})2\pi g e \quad \forall n \in \mathbb{Z} \tag{61}$$

equivalent to

$$\rho^{\alpha\beta}(0,x) = (1-\delta^{\alpha\beta})2\pi g e \delta(x). \tag{62}$$

Choosing

$$a^\alpha(t) = \sum_{k\in\mathbb{Z}} \sum_{\gamma=1}^{N} (1-\delta^{\alpha\gamma}\delta_{k,0}) \frac{-2\pi g e}{(k + g[q^\alpha(t) - q^\gamma(t)])^2} \tag{63}$$

implies $\partial_0 \hat{\rho}^{\alpha\beta}(t,n) = 0$, as can be seen by inserting Eq. (63) in Eq. (44). We thus obtain $\hat{\rho}^{\alpha\beta}(t,n) = (1-\delta^{\alpha\beta})2\pi g e$ for all t, and p^α and q^α are in fact the only dynamical variables in the system.

Inserting Eq. (61) into the Hamiltonian Eq. (23) and using Eq. (49) we obtain

$$\mathcal{H} = \frac{1}{2}\sum_{\alpha=1}^{N}(p^\alpha)^2 + \frac{e^2}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \frac{(\pi g)^2}{\sin^2(\pi g[q^\alpha - q^\beta])} \tag{64}$$

which for $g = a/\pi$ equals the Hamiltonian of the Sutherland model as discussed in Section 1.

We now show how to solve the initial value problem for the Hamiltonian equations following from the Hamiltonian Eq. (64) and the Poisson brackets $\{q^\alpha, p^\beta\} = \delta^{\alpha\beta}$. We first have to determine the initial conditions for the Yang-Mills fields corresponding to $q^\alpha(0) = q_0^\alpha$ and $p^\alpha(0) = p_0^\alpha$ all real. For $t = 0$ we can write Gauss' law Eq. (33) as

$$\partial_1\left(e^{igQ_0x}E(0,x)e^{-igQ_0x}\right) = e^{igQ_0x}\rho(0,x)e^{-igQ_0x} \tag{65}$$

where $Q_0 = Q(0) = \mathrm{diag}(q_0^1, \ldots, q_0^N)$ and $\rho(0,x)$ is given by Eq. (62). Since $\rho(0,x) = 0$ except for $x = 0$, it follows that $e^{igQ_0x}E(0,x)e^{-igQ_0x}$ is constant in the intervals $-\pi \leq x < 0$ and $0 < x < \pi$. We therefore can write

$$E(0,x) = e^{-igQ_0x}B_\pm e^{igQ_0x} \quad \text{for } x \gtrless 0 \tag{66}$$

with $B_\pm$ constant matrices.

To determine the matrices $B_\pm$ we integrate Eq. (65) from $0-\epsilon$ to $0+\epsilon$ and then take the limit $\epsilon \downarrow 0$. This gives

$$B_+^{\alpha\beta} - B_-^{\alpha\beta} = (1-\delta^{\alpha\beta})2\pi g e.$$

We also recall $\int_{-\pi}^{\pi} dx E^{\alpha\alpha}(0,x) = 2\pi p_0^\alpha$ which yields

$$\frac{1}{2}\left(B_-^{\alpha\alpha} + B_+^{\alpha\alpha}\right) = p_0^\alpha.$$

Moreover, the boundary condition $E(0,-\pi) = E(0,\pi)$ implies $e^{2i\pi g Q_0} B_- e^{-2i\pi g Q_0} = B_+$, or equivalently

$$B_-^{\alpha\beta} e^{2i\pi g(q_0^\alpha - q_0^\beta)} = B_+^{\alpha\beta}.$$

A straightforward computation shows that these equations determine the elements of the matrices B_- and B_+ as follows,

$$B_\pm^{\alpha\beta} = \delta^{\alpha\beta} p_0^\alpha + \left(1-\delta^{\alpha\beta}\right) \frac{e\pi g e^{\pm i\pi g[q_0^\alpha - q_0^\beta]}}{i\sin(\pi g[q_0^\alpha - q_0^\beta])}. \tag{67}$$

Inserting the initial condition $A_1(0,x) = Q_0$ and Eq. (66) in Eq. (47) we finally obtain the explicit solution of the Yang-Mills equation in the Weyl-gauge,

$$A_1(t,x) = e^{-igQ_0x}\left(Q_0 + B_\pm t\right) e^{igQ_0x} \quad \text{for } x \gtrless 0 \tag{68}$$

(we used $Q_0 = e^{-igQ_0x} Q_0 e^{igQ_0x}$).

We now determine the gauge transformation which transforms this solution to the diagonal Coulomb gauge. For that we need to solve Eq. (8) for $S(t,x)$. We define $\tilde{S}(t,x) = e^{igQ_0x} S(t,x)$ and observe that Eqs. (8) and (68) imply

$$\partial_1 \tilde{S}(t,x) + igB_\pm t\tilde{S}(t,x) = 0 \quad \text{for } x \gtrless 0. \tag{69}$$

This latter equation can be solved trivially,

$$\tilde{S}(t,x) = e^{-igB_\pm tx}\tilde{S}(t,0) \quad \text{for } x \gtrless 0. \tag{70}$$

We thus obtain $\tilde{S}(t,\pi) = e^{-i\pi g B_+ t}\tilde{S}(t,0)$ and $\tilde{S}(t,0) = e^{-i\pi g B_- t}\tilde{S}(t,-\pi)$, and

$$\begin{aligned} S(t,\pi) &= e^{-i\pi g Q_0}\tilde{S}(t,\pi) = e^{-i\pi g Q_0} e^{-i\pi g B_+ t}\tilde{S}(t,0) = \\ &= e^{-i\pi g Q_0} e^{-i\pi g B_+ t} e^{-i\pi g B_- t}\tilde{S}(t,-\pi) = \\ &= e^{-i\pi g Q_0} e^{-i\pi g B_+ t} e^{-i\pi g B_- t} e^{-i\pi g Q_0} \end{aligned}$$

where we used $S(t, -\pi) = I$.

In the present case we can easily prove that the Yang-Mills field $A_1(t,x)$ Eq. (68) is generic, i.e., if $q_0^\alpha \neq q_0^\beta$ for all $\alpha \neq \beta$ then particle collisions never occur. This follows from the fact that the Hamiltonian $\mathcal{H}$ in Eq. (64) is conserved under the time evolution, and this implies an upper bound on $\sin^{-2}(\pi g[q^\alpha - q^\beta])$. There exists therefore an $\epsilon > 0$ such that $|q^\alpha(t) - q^\beta(t) - n/g| > \epsilon$ for all $\alpha \neq \beta$, $t \in \mathbb{R}$, and all integers n. Our discussion in Section 2.2 implies that the latter is equivalent to $A_1(t,x)$ being generic, i.e., regular at all times t.

From the discussion in Section 4 we thus obtain the

Result 2: *The solution of the initial value problem for the Sutherland model Eq. (64) is given by the eigenvalues $\lambda^\alpha(t)$ of the matrix*

$$e^{-i\pi g Q_0} e^{-i\pi g B_+ t} e^{-i\pi g B_- t} e^{-i\pi g Q_0} \tag{71}$$

with the matrix elements of $B_\pm$ given in Eq. (67) and $Q_0 = \mathrm{diag}(q_0^1, \ldots, q_0^N)$, according to[4] *$q^\alpha(t) = -\log(\lambda^\alpha(t))/(2\pi i g)$.*

Remark: To compare this with the well-known result reviewed in [7] we observe that $B_\pm = e^{\pm i\pi g Q_0} B e^{\mp i\pi g Q_0}$, where

$$B^{\alpha\beta} = \delta^{\alpha\beta} p_0^\alpha + \left(1 - \delta^{\alpha\beta}\right) \frac{e\pi g}{i \sin(\pi g[q_0^\alpha - q_0^\beta])}.$$

Using this it is easy to see that the matrix in Eq. (71) has the same eigenvalues as the matrix

$$e^{-i\pi g Q_0} e^{-2i\pi g B t} e^{-i\pi g Q_0}, \tag{72}$$

which shows that the result above is identical with the one derived in Ref. [7].

We finally show how to recover the Lax pair and conservation laws for the Sutherland model in our formalism. We observe that our computation of $E(t,x)$ at $t = 0$ in terms of $q_0^\alpha = q^\alpha(t=0)$ and $p_0^\alpha = p^\alpha(t=0)$ immediately generalizes to $t \neq 0$. From our solution at $t = 0$ we therefore can read off that

$$E(t,x) = e^{-ig(x \mp \pi)Q(t)} B(t) e^{ig(x \mp \pi)Q(t)} \qquad \text{for } x \gtrless 0 \tag{73}$$

with

$$B^{\alpha\beta}(t) = \delta^{\alpha\beta} p^\alpha(t) + \left(1 - \delta^{\alpha\beta}\right) \frac{e\pi g}{i \sin(\pi g[q^\alpha(t) - q^\beta(t)])}. \tag{74}$$

[4] the branch of the log is fixed by continuity of $q^\alpha(t)$

Inserting Eqs. (61) and (63) in Eq. (41) and using $A_0(t,x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{inx}\hat{A}_0(t,n)$ together with the identities in Eqs. (50) and (52), we obtain after a simple computation

$$\begin{aligned} A_0^{\alpha\beta}(t,x) &= -\delta^{\alpha\beta} eg \left(\sum_{\substack{\gamma=1 \\ \gamma\neq\alpha}}^{N} \frac{\pi^2}{\sin^2(\pi g[q^\alpha(t) - q^\gamma(t)])} + \frac{\pi^2}{3} \right) \\ &+ (1-\delta^{\alpha\beta}) e g e^{-igx_{2\pi}[q^\alpha(t)-q^\beta(t)]} \times \\ &\times \left(\frac{\pi^2}{\sin^2(\pi g[q^\alpha(t) - q^\beta(t)])} + i\pi x_{2\pi} \cot(\pi g[q^\alpha(t) - q^\beta(t)]) - \pi|x_{2\pi}| \right). \end{aligned}$$

We thus have found a one-parameter family of Lax pairs for the Sutherland model. However, due to cyclicity of trace, the corresponding conservation laws are independent of x: $tr[E(t,x)^n] = tr[B(t)^n]$. It is interesting to note that the standard Lax pair for the Sutherland model, derived e.g. in Ref. [7], is recovered from our Lax pair family above by setting $x = \pi$.

4.3 Example 3: CM model with $v(r) = a^2 \sinh^{-2}(ar)$

We now make a simple but important observation: In Example 2 above we can allow for general complex initial data q_0^α and p_0^α and thus obtain time evolution equations for complex valued functions $q^\alpha(t)$ and $p^\alpha(t)$. In this case all our discussion in Example 2 above goes through, except the argument showing that no particle collisions occur. Thus our Result 2 above holds also for the general complex case at least locally in time, i.e., until the first particle collision occurs.

An important special case is obtained if e is real and all q_0^α and p_0^α are chosen purely imaginary, i.e., they are real if multiplied with i. It is easy to convince oneself that the results for this case we can obtain from the results in Example 2 simply by the following replacements

$$q^\alpha(t) \to iq^\alpha(t), \quad p^\alpha(t) \to ip^\alpha(t) \tag{75}$$

where $q^\alpha(t)$ and $p^\alpha(t)$ real. The Hamiltonian and Poisson brackets then get minus signs which can be conveniently removed by the transformation,

$$\mathcal{H} \to -\mathcal{H}, \quad \{\cdot,\cdot\} \to -\{\cdot,\cdot\}, \tag{76}$$

which leaves the Hamilton equations $\partial_0 X = \{\mathcal{H}, X\}$ invariant. The resulting initial value problem thus is equivalent to the one obtained from the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{\alpha=1}^{N}(p^\alpha)^2 + \frac{e^2}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \frac{(\pi g)^2}{\sinh^2(\pi g[q^\alpha - q^\beta])} \tag{77}$$

and the standard Poisson brackets $\{q^\alpha, p^\beta\} = \delta^{\alpha\beta}$. This initial value problem is identical with the one for the Sutherland model with the potential $v(r) = a^2 \sinh^{-2}(ar)$ and $a = \pi g$ [7]. In this case we can argue as in Example 2 that no particle collisions can occur, i.e., $|q^\alpha(t) - q^\beta(t)| > \epsilon$ for some $\epsilon > 0$ and all $t \in \mathbb{R}$ and all $\alpha \neq \beta$. Result 2 above then implies:

Result 3: *The solution of the initial value problem for the Sutherland model defined by the Hamiltonian in Eq. (77) is given by the eigenvalues $\lambda^\alpha(t)$ of the following matrix,*

$$e^{\pi g Q_0} e^{\pi g B_+ t} e^{\pi g B_- t} e^{\pi g Q_0} \tag{78}$$

with

$$B_\pm^{\alpha\beta} = \delta^{\alpha\beta} p_0^\alpha - \left(1 - \delta^{\alpha\beta}\right) \frac{e\pi g e^{\mp\pi g[q_0^\alpha - q_0^\beta]}}{\sinh(\pi g[q_0^\alpha - q_0^\beta])} \tag{79}$$

and $Q_0 = \mathrm{diag}(q_0^1, \ldots, q_0^N)$, according to $q^\alpha(t) = \log(\lambda^\alpha(t))/(2\pi g)$.

As in Example 2 one obtain Lax pair etc. and can check that what we obtain is equivalent to the known result for this model given in [7].

4.4 Example 4: Generalizing the Sutherland models

We now discuss a general case for a spin-particle dynamical system equivalent to a Yang-Mills gauge theory as discussed in Section 2 and which we can solve explicitly. Our result here generalizes the one in Ref. [1].

We choose the charge distribution of the form

$$\rho^{\alpha\beta}(t,x) \equiv \sum_{j=1}^{m} \rho_j^{\alpha\beta}(t)\delta(x - x_j) \tag{80}$$

where m is some positive integer and we assume, without loss of generality,

$$x_0 \equiv -\pi < x_1 < x_2 < \ldots < x_m < x_{m+1} \equiv \pi. \tag{81}$$

This charge distribution describes matter charges localized at the points x_j, $1 \leq j \leq m$. Using Eq. (32) it is easy to see that this ansatz is preserved under time evolution. Moreover, using

$$\hat{\rho}^{\alpha\beta}(t,n) = \sum_{k=1}^{m} \rho_k^{\alpha\beta}(t) e^{-inx_k} \tag{82}$$

we see that Eq. (21) implies the constraint

$$\sum_{j=1}^{m} \rho_j^{\alpha\alpha}(0) = 0 \quad \forall \alpha = 1, \ldots, N \tag{83}$$

on possible initial conditions for the $\rho_j^{\alpha\beta}$.

Using the identities (50) and (52), the Hamiltonian in Eq. (23) becomes (we specialize to $a^\alpha(t) = 0$ for simplicity)

$$\mathcal{H} = \frac{1}{2} \sum_{\alpha=1}^{N} (p^\alpha)^2 + \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \sum_{j,k=1}^{m} v_{jk}(q^\alpha - q^\beta) \rho_k^{\alpha\beta} \rho_j^{\beta\alpha} + \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{j,k=1}^{m} c_{jk} \rho_k^{\alpha\alpha} \rho_j^{\alpha\alpha}, \tag{84}$$

where

$$v_{jk}(r) = \frac{1}{4} e^{-igrx_{jk}} \left(\frac{1}{\sin^2(\pi g r)} + \frac{ix_{jk}}{\pi} \cot(\pi g r) - \frac{|x_{jk}|}{\pi} \right) \tag{85}$$

and

$$c_{jk} = \frac{x_{jk}^2}{8\pi^2} - \frac{|x_{jk}|}{4\pi} + \frac{1}{12}. \tag{86}$$

Here x_{jk} is defined as

$$x_{jk} = (x_j - x_k)_{2\pi}. \tag{87}$$

The Poisson brackets for this system are:

$$\{p^\alpha, q^\beta\} = \delta^{\alpha\beta} \tag{88}$$

$$\{\rho_j^{\alpha\beta}, \rho_k^{\alpha'\beta'}\} = ig2\pi\delta_{jk} \left(\delta^{\beta\alpha'} \rho_j^{\alpha\beta'} - \delta^{\beta'\alpha} \rho_j^{\alpha'\beta} \right). \tag{89}$$

We now solve the equations of motion for the system given by the Hamiltonian (84) and the Poisson brackets (88)–(89). More general than in [1], we do not assume that $q^\alpha(t)$ etc. all are *real*.

We start with Gauss' law Eq. (65) where $Q_0 = Q(0)$ is given by (7) and $\rho(0,x)$ by (80). Since $\rho(0,x) = 0$ except for $x = x_j$, we obtain from that $E(0,x) = e^{-igQ_0x} B_j e^{igQ_0x}$, when $x_j < x < x_{j+1}$, $j = 0, \ldots, m$, where B_j is some constant matrix

(we recall that $x_0 = -\pi$ and $x_{m+1} = \pi$). To determine the matrices B_j, we integrate Eq. (65) from $x_j - \epsilon$ to $x_j + \epsilon$ and then take the limit $\epsilon \downarrow 0$. This gives the recursion relations $B_j - B_{j-1} = e^{igQ_0 x_j} \rho_j(0) e^{-igQ_0 x_j}$, and the condition $E(0, -\pi) = E(0, \pi)$ implies $e^{2igQ_0\pi} B_0 e^{-2igQ_0\pi} = B_m$. Combining these equations and using the definition Eq. (42) of p^α, we obtain after a straightforward calculation,

$$\begin{aligned} B_j^{\alpha\beta} &= \delta^{\alpha\beta} \left(p^\alpha(0) + \sum_{\ell=1}^{j} \rho_\ell^{\alpha\alpha}(0) - \sum_{i=1}^{m} \frac{x_{i+1} - x_i}{2\pi} \sum_{\ell=1}^{i} \rho_\ell^{\alpha\alpha}(0) \right) + \\ &+ \left(1 - \delta^{\alpha\beta}\right) \sum_{\ell=1}^{m} \frac{\rho_\ell^{\alpha\beta}(0) e^{ig[q^\alpha(0) - q^\beta(0)][x_\ell + \pi \mathrm{sgn}(x_j - x_\ell)]}}{2i \sin(g\pi[q^\alpha(0) - q^\beta(0)])} \end{aligned} \tag{90}$$

with $\mathrm{sgn}(x) = 1$ for $x \geq 0$ and -1 for $x < 0$. With that, the solution of the Yang-Mills equations in the Weyl gauge, Eq. (47), gives

$$A_1(t, x) = e^{-igQ_0 x} \left(Q_0 + B_j t\right) e^{igQ_0 x} \text{ for } x_j < x < x_{j+1}. \tag{91}$$

Next, we solve $\partial_1 S(t, x) + igA_1(t, x) S(t, x) = 0$ which is equivalent to

$$\partial_1 \tilde{S}(t, x) + igB_j t \tilde{S}(t, x) = 0 \quad \text{ for } x_j < x < x_{j+1} \tag{92}$$

for $\tilde{S}(t, x) = e^{igQ_0 x} S(t, x)$. This implies $\tilde{S}(t, x) = e^{-igB_j t(x - x_j)} \tilde{S}(t, x_j)$ for $x_j < x < x_{j+1}$, thus

$$\begin{aligned} S(t, x_{j+1}) &= e^{-igQ_0 x_{j+1}} e^{-igB_j t(x_{j+1} - x_j)} \tilde{S}(t, x_j) = \ldots = \\ &= e^{-igQ_0 x_{j+1}} e^{-igB_j t(x_{j+1} - x_j)} e^{-igB_{j-1} t(x_j - x_{j-1})} \cdots e^{-igB_0 t(x_1 + \pi)} e^{-igQ_0 \pi}, \end{aligned} \tag{93}$$

where we used $S(t, -\pi) = I$. Especially (for $j = m$),

$$\begin{aligned} S(t, \pi) = e^{-igQ_0\pi} e^{-igB_m t(\pi - x_m)} \cdots \\ \times e^{-igB_{m-1} t(x_m - x_{m-1})} \cdots e^{-igB_0 t(x_1 + \pi)} e^{-igQ_0 \pi}. \end{aligned} \tag{94}$$

From our discussion in Section 4 we can obtain the $q^\alpha(t)$ from the eigenvalues of this matrix. Moreover,

$$\rho_j(t) = U(t, x_j)^{-1} \rho_j(0) U(t, x_j),$$

where $U(t, x_j)$ is given by (11). We thus obtain the

Result 4: *Provided no particle collisions occur in the time interval* $[0, t]$*, the solution* $q^\alpha(t)$ *of the initial value problem defined by the Hamiltonian (84) and the Poisson*

brackets (88)–(89) is given by the eigenvalues $\lambda^\alpha(t)$ of the matrix $S(t,\pi)$ defined in Eqs. (94) and (90), according to[5] $q^\alpha(t) = -\log(\lambda^\alpha(t))/(2\pi i g)$. *Moreover, the time evolution of the spin degrees of freedom is given by*

$$\rho_j(t) = e^{-ig(x_j+\pi)Q(t)} V(t)^{-1} S(t,x_j)^{-1} \rho_j(0) S(t,x_j) V(t) e^{ig(x_j+\pi)Q(t)} \tag{95}$$

where $S(t,x_j)$ is defined in Eqs. (93) and (90), $Q(t) = \mathrm{diag}(q_1(t), q_2(t), \ldots q_N(t))$ and $V(t)$ is the matrix diagonalizing $S(t,\pi)$, i.e.

$$V(t)^{-1} S(t,\pi) V(t) = e^{-ig2\pi Q(t)}.$$

As discussed in the beginning of this section, $tr[E(t,x)^n]$ is conserved under time evolution. For the special choice of charges studied above, we can evaluate $E(t,x)$ and obtain $E(t,x) = e^{-igQ(t)x} B_j(t) e^{igQ(t)x}$ for $x_j < x < x_{j+1}$ where $Q(t)$ and $B_j(t)$ are as in Eqs. (7) and (90) but with $q^\alpha(0)$, $p^\alpha(0)$, and $\rho_j^{\alpha\beta}(0)$ replaced by $q^\alpha(t)$, $p^\alpha(t)$, and $\rho_j^{\alpha\beta}(t)$, i.e., the solution of the initial value problem which we solved above. Using cyclicity of the trace, we conclude that $tr[B_j(t)^n]$, for an arbitrary positive integer n and $j = 1, \ldots m$, are time independent: Each of them is a conservation law.

The one-parameter family of Lax equations is $\partial_0 E(t,x) + ig[A_0(t,x), E(t,x)] = 0$. We now compute the corresponding family of Lax pairs explicitly, similarly as in Section 4.2. We obtain after a straightforward calculation,

$$\begin{aligned} E(t,x) &= \delta^{\alpha\beta} \left(p^\alpha(t) + \sum_{\ell=1}^{j} \rho_\ell^{\alpha\alpha}(t) - \sum_{i=1}^{m} \frac{x_{i+1} - x_i}{2\pi} \sum_{\ell=1}^{i} \rho_\ell^{\alpha\alpha}(t) \right) + \\ &+ \left(1 - \delta^{\alpha\beta}\right) e^{-igx[q^\alpha(t) - q^\beta(t)]} \sum_{\ell=1}^{m} \frac{\rho_\ell^{\alpha\beta}(t) e^{ig[q^\alpha(t)-q^\beta(t)][x_\ell + \pi \mathrm{sgn}(x_j - x_\ell)]}}{2i \sin(g\pi[q^\alpha(t) - q^\beta(t)])} \end{aligned} \tag{96}$$

and

$$\begin{aligned} A_0^{\alpha\beta}(t,x) &= -\delta^{\alpha\beta} \frac{1}{2\pi} \sum_{k=1}^{m} \rho_k^{\alpha\alpha}(t) \left(\frac{(x - x_k)_{2\pi}^2}{2} - \pi|(x - x_k)_{2\pi}| + \frac{\pi^3}{3} \right) + \\ &+ (1 - \delta^{\alpha\beta}) \frac{\pi}{2} \sum_{k=1}^{m} \rho_k^{\alpha\beta}(t) e^{-ig(x-x_k)_{2\pi}[q^\alpha(t) - q^\beta(t)]} \left(\frac{1}{\sin^2(\pi g[q^\alpha(t) - q^\beta(t)])} + \right. \\ &+ \left. \frac{i(x - x_k)_{2\pi}}{\pi} \cot(\pi g[q^\alpha(t) - q^\beta(t)]) - \frac{|(x - x_k)_{2\pi}|}{\pi} \right). \end{aligned}$$

[5] the branch of the log is fixed by continuity of $q^\alpha(t)$

Note that for $m = 1$, the dynamics of the spin and the particles decouple, and we recover the results in Sections 4.2 and 4.3.

Note that all variables $q^\alpha(t)$, $p^\alpha(t)$, and $\rho_j^{\alpha\beta}(t)$ above were allowed to be complex. If we restrict them to be real we recover the results in [1]. This generalizes the results for the Sutherland model is Section 4.2. If we assume that all $q^\alpha(t)$ and $p^\alpha(t)$ are purely imaginary and the $\rho_j^{\alpha\beta}(t)$ all real we obtain results generalizing the ones in Section 4.3 (similarly as discussed in Section 4.3, they can be obtained from the ones for the real case by the substitutions in Eqs. (75) and (76).

4.5 Example 5: Novel systems

The previous examples all where such that it was possible to give very explicit solutions. As a final example we present a case where the solution can be only described in a somewhat implicit way. It corresponds to novel integrable spin-particle systems which generalizes the model discussed in Section 4.1. The arguments here are similar to the ones in Section 4.4 and therefore we are rather brief.

We chose the charge distribution as follows

$$\rho^{\alpha\beta}(t,x) \equiv \frac{1}{\Delta_j} e^{-ig[q^\alpha(t)-q^\beta(t)]x} s_j^{\alpha\beta}(t), \quad x_{j-1} \leq x < x_j \tag{97}$$

where $\Delta_j \equiv (x_j - x_{j-1})$, $j = 1, \ldots, m$, and

$$x_0 \equiv -\pi < x_1 < x_2 < \ldots < x_{m-1} < x_m \equiv \pi \tag{98}$$

with m some positive integer. This ansatz describes (essentially) piecewise constant matter sources and is preserved under time evolution (cf. Eq. (32)). Moreover, using

$$\hat{\rho}^{\alpha\beta}(t,n) = \sum_{j=1}^{m} s_j^{\alpha\beta}(t) \frac{1}{\Delta_j} \int_{x_{j-1}}^{x_j} dx\, e^{-ig[q^\alpha(t)-q^\beta(t)]x} e^{-inx} \tag{99}$$

we see that Eq. (21) implies the following constraint

$$\sum_{j=1}^{m} s_j^{\alpha\alpha}(0) = 0 \quad \forall \alpha = 1, \ldots, N \tag{100}$$

on possible initial conditions for the $s_j^{\alpha\beta}$.

Using the identities (50) and (52), the Hamiltonian (23) becomes (again we set $a^{\alpha}(t)=0$ for simplicity)

$$\mathcal{H}=\frac{1}{2}\sum_{\alpha=1}^{N}(p^{\alpha})^{2}+\frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\sum_{j,k=1}^{m}w_{jk}(q^{\alpha}-q^{\beta})s_{k}^{\alpha\beta}s_{j}^{\beta\alpha}+\frac{1}{2}\sum_{\alpha=1}^{N}\sum_{j,k=1}^{m}d_{jk}s_{k}^{\alpha\alpha}s_{j}^{\alpha\alpha} \tag{101}$$

where

$$\begin{aligned} w_{jk}(r) &= \frac{1}{\Delta_j\Delta_k}\int_{x_{j-1}}^{x_j}dx\int_{x_{k-1}}^{x_k}dy\,\frac{1}{4}e^{-igr[(x-y)_{2\pi}-(x-y)]}\times \\ &\times\left(\frac{1}{\sin^2(\pi gr)}+\frac{i(x-y)_{2\pi}}{\pi}\cot(\pi gr)-\frac{|(x-y)_{2\pi}|}{\pi}\right) \end{aligned} \tag{102}$$

and

$$d_{jk}=\frac{1}{\Delta_j\Delta_k}\int_{x_{j-1}}^{x_j}dx\int_{x_{k-1}}^{x_k}dy\left(\frac{(x-y)_{2\pi}^2}{8\pi^2}-\frac{|(x-y)_{2\pi}|}{4\pi}+\frac{1}{12}\right) \tag{103}$$

(of course these integrals could be further evaluated). The Poisson brackets for this system are:

$$\{p^{\alpha},q^{\beta}\} = \delta^{\alpha\beta} \tag{104}$$

$$\{s_j^{\alpha\beta},s_k^{\alpha'\beta'}\} = ig2\pi\delta_{jk}\left(\delta^{\beta\alpha'}s_j^{\alpha\beta'}-\delta^{\beta'\alpha}s_j^{\alpha'\beta}\right) \tag{105}$$

where the last equation is obtained by inserting

$$s_j^{\alpha\beta}(t)=\int_{x_{j-1}}^{x_j}dx\,e^{ig[q^{\alpha}(t)-q^{\beta}(t)]x}\rho^{\alpha\beta}(t,x) \tag{106}$$

in Eq. (18).

We now show how to solve the initial value problem for the Hamiltonian equations of this model. Writing Gauss' law in the form Eq. (65) and introducing ($Q_0=Q(0)$, as in the previous Section) $B(0,x)=e^{igQ_0x}E(0,x)e^{-igQ_0x}$ we obtain with Eq. (97),

$$B^{\alpha\beta}(0,x)=C_j^{\alpha\beta}(0)+\frac{(x-x_{j-1})}{(x_j-x_{j-1})}s_j^{\alpha\beta}(0)\quad\text{for } x_{j-1}\leq x<x_j \tag{107}$$

where

$$C_j^{\alpha\beta}(0)=B^{\alpha\beta}(0,-\pi)+\sum_{k=1}^{j-1}s_k^{\alpha\beta}(0). \tag{108}$$

Similarly as in Section 4.4, the conditions $2\pi p^\alpha(0) = \int_{-\pi}^{\pi} dx\, E^{\alpha\alpha}(0,x)$ and $E(0,\pi) = E(0,-\pi)$ determine $B(0,-\pi)$, and we obtain after some computations,

$$\begin{aligned} B^{\alpha\beta}(0,-\pi) &= \delta^{\alpha\beta}\left(p^\alpha(0) - \sum_{j=1}^{m} \frac{\Delta_j}{2\pi}\left(\tfrac{1}{2}s_j^{\alpha\alpha}(0) + \sum_{k=1}^{j-1} s_k^{\alpha\alpha}(0)\right)\right) + \\ &+ \left(1-\delta^{\alpha\beta}\right)\frac{e^{-ig[q^\alpha(0)-q^\beta(0)]\pi}}{2i\sin(\pi g[q^\alpha(0)-q^\beta(0)])}\sum_{j=1}^{m} s_j^{\alpha\beta}(0). \end{aligned} \tag{109}$$

We thus obtain $A_1(t,x) = e^{-igQ_0x}\,[Q_0 + B(0,x)]\,e^{igQ_0x}$.

To find the gauge transformation bringing this to the diagonal Coulomb gauge we define $\tilde{S}(t,x) = e^{igQ_0x}S(t,x)$ converting the differential equation in (8) to $\partial_1\tilde{S}(t,x) + igB(0,x)t\tilde{S}(t,x)$. Requiring $S(t,-\pi) = I$ we obtain

$$S(t,x) = e^{-igQ_0x}T_j\left(t, \tfrac{(x-x_{j-1})}{(x_j-x_{j-1})}\right)T_{j-1}(t,1)\cdots T_1(t,1)e^{-igQ_0\pi} \quad \text{for } x_{j-1} \le x < x_j \tag{110}$$

where

$$T_j(t,r) = \mathcal{P}\exp\left(-igt\Delta_j \int_0^r d\xi[C_j(0) + \xi s_j(0)]\right). \tag{111}$$

This gives the solution of the model: The $q^\alpha(t)$ are determined by the eigenvalues of $S(t,\pi)$ as in the previous examples. To compute $s_j(t)$ is more complicated here, however: One can use Eq. (106) with $\rho(t,x) = U^{-1}(t,x)\rho(0,x)U(t,x)$ and $U(t,x)$ determined by $S(t,x)$ as discussed in Section 2.2. In the present case we could not make the solution more explicit.

It is possible to write down explicitly the Lax pair family for the present model, and there is also a 1-parameter family of conservation laws here, namely $tr[B(t,x)^n]$, $n \in \mathbb{N}$ and $x \in [-\pi,\pi]$, where $B(t,x)$ is as in Eqs. (107)–(109) with $q^\alpha(0)$, $p^\alpha(0)$ and $s_j(0)$ replaced by $q^\alpha(t)$, $p^\alpha(t)$ and $s_j(t)$.

A Appendix: Proofs

In this Appendix we prove the identities Eqs. (50) and (52). To prove Eq. (50) we define the following function,

$$g(r,s) := \sum_{n\in\mathbb{Z}} \frac{e^{i(n+r)s}}{(n+r)^2} \tag{112}$$

(r is a complex and s a variable). We observe

$$\frac{\partial^2 g(r,s)}{\partial s^2} = -\sum_{n\in\mathbf{Z}} e^{i(n+r)s} = -2\pi\delta(s) \quad \text{for } -\pi \le s \le \pi.$$

Integrating this twice gives

$$g(r,s) = A(r) + B(r)s - \pi|s| \quad \text{for } -\pi \le s \le \pi \tag{113}$$

where $A(r)$ and $B(r)$ are integration constants. We now determine $A(r)$ and $B(r)$. Using Eq. (49) we conclude that

$$g(r,0) = A(r) = \sum_{n\in\mathbf{Z}} \frac{1}{(n+r)^2} = \frac{\pi^2}{\sin^2 \pi r}.$$

Moreover, the definition Eq. (112) of $g(r,s)$ implies

$$g(r,-\pi)e^{ir\pi} = g(r,\pi)e^{-ir\pi}.$$

Inserting Eq. (113) we get

$$B(r) = i\pi \cot(\pi r).$$

We thus obtain,

$$\sum_{n\in\mathbb{Z}} \frac{e^{ins}}{(n+r)^2} = e^{-irs}\left(\frac{\pi^2}{\sin^2 \pi r} + i\pi s \cot(\pi r) - \pi|s|\right) \quad \text{for } -\pi \le s \le \pi. \tag{114}$$

The l.h.s. of this equation obviously is periodic in s with period 2π. To extend the r.h.s. to all real s we therefore only need to replace s by $s_{2\pi} := s - 2\pi n$ with the integer n chosen such that $-\pi \le s - 2\pi n < \pi$. This proves Eq. (50).

To derive Eq. (52) we define

$$h(s) := \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e^{ins}}{n^2} \tag{115}$$

(s is real) and compute

$$\frac{d^2 h(s)}{ds^2} = -\sum_{\substack{n\in\mathbf{Z}\\ n\neq 0}} e^{ins} = -2\pi\delta(s) + 1 \quad \text{for } -\pi \le s \le \pi.$$

Integrating this twice yields

$$h(s) = -\pi|s| + \frac{s^2}{2} + Cs + D$$

where C and D are integration constants. Periodicity of $h(s)$ implies $C = 0$, and the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(see, e.g., [12]) determines $D = \frac{\pi^2}{3}$. Thus

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{e^{ins}}{n^2} = -\pi|s| + \frac{s^2}{2} + \frac{\pi^2}{3} \quad \text{for } -\pi \leq s \leq \pi. \tag{116}$$

Again the r.h.s. can be extended to all real s by replacing s by $s_{2\pi}$ as defined above. This proves Eq. (52).

B Appendix: Poisson bracket for the charges

In this appendix we derive the Poisson bracket for the charges $\rho(x)$, i.e., Eq. (18). As mentioned in the text, consistency of Gauss' law $G(x) \simeq 0$ requires

$$\partial_0 G^{\alpha'\beta'}(y) = \{G^{\alpha'\beta'}(y), H\} \simeq 0 \quad \forall \alpha', \beta' = 1, \ldots, N. \tag{117}$$

Inserting Eqs. (15) and (17) we can rewrite $\{G^{\alpha'\beta'}(y), H\} \simeq 0$ as

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \sum_{\alpha,\beta=1}^{N} A_0^{\beta\alpha}(x) \left\{\rho^{\alpha\beta}(x), \rho^{\alpha'\beta'}(y)\right\} \simeq \\
\simeq \; & \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \sum_{\alpha,\beta,\gamma=1}^{N} ig \left(\tfrac{1}{2} \left\{ E^{\beta\alpha}(x)E^{\alpha\beta}(x), A_1^{\alpha'\gamma}(y)E^{\gamma\beta'}(y) - E^{\alpha'\gamma}(y)A_1^{\gamma\beta'}(y)\right\} \right. \\
- \; & A_0^{\beta\alpha}(x) \left\{ \partial_x E^{\alpha\beta}(x), A_1^{\alpha'\gamma}(y)E^{\gamma\beta'}(y) - E^{\alpha'\gamma}(y)A_1^{\gamma\beta'}(y)\right\} \\
- \; & A_0^{\beta\alpha}(x) \left\{ A_1^{\alpha\gamma}(x)E^{\gamma\beta}(x) - E^{\alpha\gamma}(x)A_1^{\gamma\beta}(x), \partial_y E^{\alpha'\beta'}(y)\right\} - igA_0^{\beta\alpha}(x) \\
& \left. \times \left\{ A_1^{\alpha\gamma}(x)E^{\gamma\beta}(x) - E^{\alpha\gamma}(x)A_1^{\gamma\beta}(x), A_1^{\alpha'\gamma}(y)E^{\gamma\beta'}(y) - E^{\alpha'\gamma}(y)A_1^{\gamma\beta'}(y)\right\} \right).
\end{aligned}$$

Here we used that the only non-vanishing Poisson brackets are those for A_1 with E and for ρ with itself. Using now the Poisson bracket (16) the r.h.s. of this becomes

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \sum_{\alpha,\beta=1}^{N} A_0^{\beta\alpha}(x)\{\rho^{\alpha\beta}(x), \rho^{\alpha'\beta'}(y)\} = \\
= \;& 0 + ig\int_{-\pi}^{\pi} dx \sum_{\gamma=1}^{N}\Big(A_0^{\alpha'\gamma}(x)E^{\gamma\beta'}(y)\partial_x\delta(x-y) - A_0^{\gamma\beta'}(x)E^{\alpha'\gamma}(y)\partial_x\delta(x-y) \\
+ \;& A_0^{\gamma\beta'}(x)E^{\alpha'\gamma}(x)\partial_x\delta(x-y) - A_0^{\alpha'\gamma}(x)E^{\gamma\beta'}(x)\partial_x\delta(x-y)\Big) \\
- \;& (ig)^2\int_{-\pi}^{\pi} dx\, \delta(x-y)\Big(\left[\left[A_1(y), A_0(y)\right], E(y)\right]^{\alpha'\beta'} + \left[\left[A_0(y), E(y)\right], A_1(y)\right]^{\alpha'\beta'}\Big) \\
= \;& ig\int_{-\pi}^{\pi} dx\, \delta(x-y)\Big(-\left[\partial_1 E(y), A_0(x)\right]^{\alpha'\beta'} - ig\left[\left[A_1(y), E(y)\right], A_0(x)\right]^{\alpha'\beta'}\Big) \\
= \;& (-ig)\int_{-\pi}^{\pi} dx\, \delta(x-y)\left[\partial_1 E(y) + ig\left[A_1(y), E(y)\right], A_0(x)\right]^{\alpha'\beta'} \\
\simeq \;& (-ig)\int_{-\pi}^{\pi} dx\, \delta(x-y)\left[\rho(y), A_0(y)\right]^{\alpha'\beta'} \\
= \;& \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \sum_{\alpha,\beta=1}^{N} A_0^{\beta\alpha}(y)\Big(ig2\pi\left[\rho^{\alpha\beta'}(y)\delta^{\alpha'\beta} - \rho^{\alpha'\beta}(y)\delta^{\alpha\beta'}\right]\delta(x-y)\Big).
\end{aligned}
$$

Here the second equality follows by using partial integration and the Jacobi identity, and in the fourth equality Gauss' law was used. From this the Poisson brackets in Eq. (18) follow.

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